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SOME REPRESENTATIONS OF SUBGROUPS OF THE MAPPING CLASS GROUPS OF SURFACES AND SECONDARY INVARIANTS

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1. INTRODUCTION

Let Σ_g be a closed oriented surface of genus $g \geqq 1$ and $\mathcal{M}_g$ its mapping class group consisting of the isotopy classes of orientation preserving diffeomorphisms of Σ_g. We denote the 2-sphere with 3-holes by P. For any $a, b \in \mathcal{M}_g$, let $N_{a,b}$ be the Σ_g-bundle over P with monodromies a^{-1} and b^{-1}.

Meyer's signature 2-cocycle

$$sign_g \colon \mathcal{M}_g \times \mathcal{M}_g \to \mathbb{Z}$$

is defined by $sign_g(a,b) \colon = sign(N_{a,b})$, where $sign(N_{a,b})$ is the signature of 4-manifold $N_{a,b}$ (see [10, 1]). Novikov additivity for the signature of manifolds shows that $sign_g$ satisfies the cocycle condition. Meyer also defined a 2-cocycle τ_g on $Sp(2g, \mathbb{Z})$ over $\mathbb{Z}$, which is also called signature 2-cocycle. It is well-known that the equality $sign_g = \zeta_g^* \tau_g$ holds, where ζ_g is the standard representation of $\mathcal{M}_g$ to $Sp(2g, \mathbb{Z})$ induced from the obvious action of $\mathcal{M}_g$ on the first cohomology group of Σ_g.

Let ι be the hyperelliptic involution on Σ_g depicted in Figure 1.

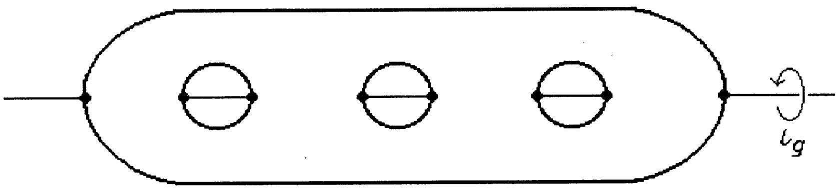

FIGURE 1. The hyperelliptic involution ι on Σ_g.

The hyperelliptic mapping class group $\mathcal{H}_g$ of Σ_g is the subgroup of $\mathcal{M}_g$ consisting of elements which commute with the class of ι. It is known that $\mathcal{M}_1 = \mathcal{H}_1 = SL(2, \mathbb{Z})$, $\mathcal{M}_2 = \mathcal{H}_2$ and that $\mathcal{H}_g (g \geqq 3)$ is a subgroup of $\mathcal{M}_g$ of infinite index.

Meyer's signature cocycle $sign_g$ defines a nontrivial class of the second cohomology group of $\mathcal{M}_g$ with coefficients in $\mathbb{Z}$ and its restriction to $\mathcal{H}_g$ is also nontrivial. But it is trivial in the cohomology group of $\mathcal{H}_g$ with coefficients in $\mathbb{Q}$. Thus there exists a function or a 1-cochain

$$\phi_g \colon \mathcal{H}_g \to \mathbb{Q}$$

such that $sign_g = \delta\phi_g$, where δ denotes the coboundary operator defined by $\delta\phi_g(a,b) = \phi_g(b) - \phi_g(ab) + \phi_g(a)$ for $a, b \in \mathcal{H}_g$. It follows that ϕ_g is unique from the fact that the first cohomology group of $\mathcal{H}_g$ vanishes. This function ϕ_g is called Meyer function. It is known that it is conjugacy invariant. Its values are contained in $\frac{1}{2g+1}\mathbb{Z}$ and concrete values on Lickorish generators and BSCC maps are calculated by Endo [4], Matsumoto [9] and Morifuji [11].

In the case of $g = 1$, under the identification $\mathcal{M}_1 \cong \mathcal{H}_1 \cong SL(2,\mathbb{Z})$, Meyer [10] and Atiyah [1] gave the explicit expression of the Meyer function using the Dedekind sums (see also [7]). Thus we can compute the values of it. Moreover Atiyah [1] put various geometric interpretations on the values of ϕ_1 on hyperbolic elements. Hereafter we regard $SL(2,\mathbb{Z})(= Sp(2,\mathbb{Z}))$ as the domain of ϕ_1. Hence we have $\delta\phi_1 = \tau_1$.

In this paper we study some representations induced from the actions of subgroups of the mapping class groups of a surface on the first cohomology group of $\pi_1(\Sigma_g)$ with coefficients in the module obtained from the nontrivial representation of $\pi_1(\Sigma_g)$ to $\mathbb{Z}_2 = Aut(\mathbb{Z})$. As an application of them, in the case of $g = 1, 2$ (see also [5, 6]) and 3, we define some functions on subgroups of $\mathcal{H}_g$ using Atiyah-Patodi-Singer ρ-invariants and state that the difference of our function from the Meyer function is a nontrivial homomorphism on the subgroup. Moreover we state that the Meyer function coincides with the average of our functions on a certain subgroup.

2. Some representations of subgroups of the mapping class groups

Let Σ_g be a closed oriented surface of genus $g \geqq 1$ and $* \in \Sigma_g$ a base point. Let $\omega \colon \pi_1(\Sigma_g, *) \to \mathbb{Z}_2$ be a nontrivial homomorphism which is also regarded as an element of $H^1(\Sigma_g; \mathbb{Z}_2)$. If we regard $\mathbb{Z}_2$ as $Aut(\mathbb{Z})$, then using ω, we can obtain

$\pi_1(\Sigma_g, *)$-module $\mathbb{Z}$, which is denoted by $\mathbb{Z}_\omega$. We consider the first cohomology group $H^1(\pi_1(\Sigma_g, *), \mathbb{Z}_\omega)$ which is isomorphic to $\mathbb{Z}^{2(g-1)} \oplus \mathbb{Z}_2$. Moreover it has a natural pairing defined by the cup product, the pairing $\mathbb{Z}_\omega \otimes \mathbb{Z}_\omega \cong \mathbb{Z}$ and the evaluation on the fundamental class of Σ_g. It is found that this pairing induces a symplectic form on the quotient group $H^1(\pi_1(\Sigma_g, *), \mathbb{Z}_\omega)/torsion$ and that it is isomorphic to the standard one on $\mathbb{Z}^{2(g-1)}$.

Let $\mathcal{M}_{g*}$ be the mapping class group of Σ_g with a base point and $\mathcal{M}_{g*}^\omega$ the subgroup of it consisting of elements which preserve ω. This subgroup acts on the group $H^1(\pi_1(\Sigma_g, *), \mathbb{Z}_\omega)/torsion$ by pullback. Since this action preserves the symplectic form, if we take a symplectic basis for it, we have the representation

$$\zeta_{g*}^\omega \colon \mathcal{M}_{g*}^\omega \to Sp(2(g-1), \mathbb{Z}).$$

These representations are related to prym representations of Looijenga [8]. Some properties of ζ_{g*}^ω were investigated in [5, 6].

In this section we study the restrictions of them to subgroups of the hyperelliptic mapping class group of genus $g \geqq 3$.

The hyperelliptic mapping class group $\mathcal{H}_g$ of Σ_g is naturally isomorphic to the group of isotopy classes of orientation preserving diffeomorphisms which commute with ι under isotopy which also commutes with ι [3]. This description of $\mathcal{H}_g$ shows that it acts the set of the fixed points of ι. Thus we have the representation $\sigma \colon \mathcal{H}_g \to \mathfrak{S}_{2g+2}$, where $\mathfrak{S}_{2g+2}$ denotes the symmetric group of degree $2g+2$ which is the number of the fixed points of ι. Let $\mathcal{H}_g^\sigma$ be the kernel of the representation of σ. Let $j \colon \mathcal{M}_{g*} \to \mathcal{M}_g$ be the natural homomorphism, then we have the short exact sequence $1 \to \pi_1(\Sigma_g, *) \to \mathcal{M}_{g*} \xrightarrow{j} \mathcal{M}_g \to 1$. Put $\mathcal{H}_{g*} = j^{-1}(\mathcal{H}_g)$ and $\mathcal{H}_{g*}^\sigma = j^{-1}(\mathcal{H}_g^\sigma)$. The following lemma is known.

Lemma 1. *For any $a \in \mathcal{H}_g^\sigma$, the induced homomorphism a^* on $H^1(\Sigma_g; \mathbb{Z}_2)$ is the identity.*

By this lemma, we have $\mathcal{H}_g^\sigma \subset \mathcal{M}_g^\omega$ and $\mathcal{H}_{g*}^\sigma \subset \mathcal{M}_{g*}^\omega$ for any $\omega \neq 0 \in H^1(\Sigma_g; \mathbb{Z}_2)$. We denote the class of ι in $\mathcal{H}_g^\sigma$ by the same letter ι.

Lemma 2. *For any lift $\tilde{\iota}$ of $\iota \in \mathcal{H}_g^\sigma$ to $\mathcal{H}_{g*}^\sigma$, the image of $\tilde{\iota}$ by ζ_{g*}^ω commutes with those of all elements of $\mathcal{H}_{g*}^\sigma$.*

The fundamental group $\pi_1(\Sigma_g, *)$ of Σ_g is presented by $< \alpha_i, \beta_i \ (1 \leqq i \leqq g) \mid \prod_{i=1}^g [\alpha_i, \beta_i] = 1 >$, where the generators are depicted in Figure 2.

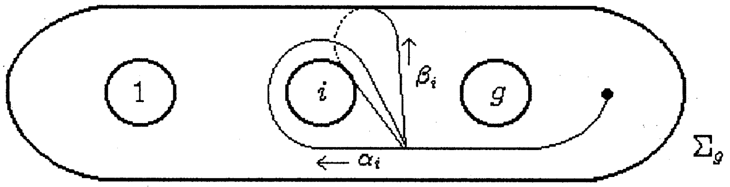

FIGURE 2. The generators of $\pi_1(\Sigma_g, *)$.

Let α_i^*, β_i^* $(1 \leqq i \leqq g)$ be the dual basis for $H^1(\Sigma_g; \mathbb{Z}_2)$ to the one for $H_1(\Sigma_g; \mathbb{Z}_2)$ which is given by the homology classes of α_i, β_i.

Lemma 3. *For any nonzero class $\omega \in H^1(\Sigma_g; \mathbb{Z}_2)$, there exists $a \in \mathcal{H}_g$ such that $a^*\omega = \alpha_k^*$ for some k.*

Direct computations show that the representation matrix of $\zeta_{g*}^{\alpha_k^*}(\tilde{\iota})$ with respect to a symplectic basis for $H^1(\pi_1(\Sigma_g, *), \mathbb{Z}_\omega)/torsion$ is given by $\pm(I_{2(k-1)} \oplus (-I_{2(g-k)}))$, where $I_{2(k-1)}$ and $I_{2(g-k)}$ are the identity matrices of rank $2(k-1)$ and $2(g-k)$ respectively. And $H^1(\pi_1(\Sigma_g, *), \mathbb{Z}_\omega)/torsion$ decomposes to the direct sum of two symplectic submodules over $\mathbb{Z}$. This result and Lemma 2 imply the following lemma.

Lemma 4. *For any nonzero $\omega \in H^1(\Sigma_g; \mathbb{Z}_2)$, the representation matrix of $\zeta_{g*}^\omega(\tilde{\iota})$ with respect to some symplectic basis is $\pm(I_{2(k-1)} \oplus (-I_{2(g-k)}))$ for some k. Moreover $H^1(\pi_1(\Sigma_g, *), \mathbb{Z}_\omega)/torsion$ is decomposed to the direct sum of two symplectic submodules over $\mathbb{Z}$ on which $\zeta_{g*}^\omega(\tilde{\iota})$ is $\pm$ the identity.*

If we take a fixed point e of ι as a base point $*$ of Σ_g, we can consider the group $\mathcal{H}_g^\sigma$ as a subgroup of $\mathcal{H}_{g*}^\sigma$.

Corollary 5. *The representation ζ_{ge}^ω induces two representations of $\mathcal{H}_g^\sigma$ to $Sp(2(k-1), \mathbb{Z})$ and $Sp(2(g-k).\mathbb{Z})$, where k is the integer in Lemma 3.*

3. Some functions on subgroups of $\mathcal{H}_{g*}$ of low genus.

In this section we consider the case of $g = 1, 2$ and 3.

Let H' be the set $H^1(\Sigma_g; \mathbb{Z}_2) \setminus \{0\}$ for $g = 1, 2$ and the set $\{\omega \in H^1(\Sigma_g; \mathbb{Z}_2) \setminus \{0\} |\ k = 2 \text{ in Lemma 4}\}$ for $g = 3$.

Lemma 6. *The number $\sharp H'$ of the elements of H' is 3, 15 and 35 for $g = 1, 2$ and 3 respectively.*

For each $\omega \in H'$, let Δ^{ω}_{g*} denote $\mathcal{H}_{g*} \cap \mathcal{M}^{\omega}_{g*}$ for $g = 1, 2$ and $\mathcal{H}^{\sigma}_{g*}$ for $g = 3$. For any $\omega \in H'$, the image of Δ^{ω}_{g*} by ζ^{ω}_{g*} is contained in $\{id\}$, $SL(2, \mathbb{Z})$ and $SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z})$ for $g = 1, 2$ and 3 respectively under an appropriate choice of a symplectic basis for the representation space. In the case of $g = 3$, let $\zeta^{\omega+}_{g*}$ and $\zeta^{\omega-}_{g*}$ be the composition of ζ^{ω}_{g*} with the projection from $SL(2, \mathbb{Z})$ to the first and second factor $SL(2, \mathbb{Z})$ respectively.

For each $\omega \in H'$, the function

$$\Phi^{\omega}_{g*} \colon \Delta^{\omega}_{g*} \to \frac{1}{3}\mathbb{Z}$$

is defined by 0, $(\zeta^{\omega}_{2*})^* \phi_1$ and $(\zeta^{\omega+}_{3*})^* \phi_1 + (\zeta^{\omega-}_{3*})^* \phi_1$ for $g = 1, 2$ and 3 respectively. It is easy to see that these functions are well defined.

Lemma 7. *The equality $\delta \Phi^{\omega}_{g*} = (\zeta^{\omega}_{g*})^* \tau_{g-1}$ holds on Δ^{ω}_{g*} for each $\omega \in H'$.*

4. The main theorem

In this section we define some functions on subgroups of the mapping class groups and state the main theorem.

Let ω be a nonzero class in $H^1(\Sigma_g; \mathbb{Z}_2)$. For any $a \in \mathcal{M}^{\omega}_{g*}$, put $M_a := \Sigma_g \times [0, 1]/(x, 0) \sim (a(x), 1)$. Then M_a is a Σ_g-bundle over $S^1 = [0, 1]/0 \sim 1$ with the identification i of Σ_g with the fiber at $0 \in S^1$ and with the section $s \colon S^1 \to M_a$ defined by the base point $*$ of Σ_g. It is easily checked that there is a unique homomorphism $\omega_a \colon \pi_1(M_a, s(0)) \to \mathbb{Z}_2 = \{\pm 1\} \subset U(1)$ satisfying the equalities $i^* \omega_a = \omega$ and $s^* \omega_a = 1$. We define the function $\rho_\omega \colon \mathcal{M}^{\omega}_{g*} \to \mathbb{Q}$ by $\rho_\omega(a) \colon = \rho_{\omega_a}(M_a)$ for each $a \in \mathcal{M}^{\omega}_{g*}$. Here $\rho_{\omega_a}(M_a)$ is the Atiyah-Patodi-Singer ρ-invariant for (M_a, ω_a).

In general, the Atiyah-Patodi-Singer ρ-invariant is a diffeomorphism invariant for a pair of a closed oriented manifold of odd dimension and a unitary representation of the fundamental group of it to $U(n)$. If a metric on the manifold is given, then the invariant is defined by the difference of the η-invariant of the signature operator on the manifold and n times that of signature operator with coefficients in the flat bundle obtained from the unitary representation. Thus ρ-invarinats take their values in $\mathbb{R}$. If a unitary representation factors through a finite group, then the value of the ρ-invariant belongs to $\mathbb{Q}$.

For each $\omega \in H'$, we define a rational valued function μ^{ω}_{g*} on Δ^{ω}_{g*} by

$$\mu^{\omega}_{g*} \colon = \rho_{\omega} + \Phi^{\omega}_{g*}.$$

These functions have the following properties.

Lemma 8. *For any $a \in \Delta^{\omega}_{g*}$ and $f \in \mathcal{H}_{g*}$, the following hold.*

1. $\mu^{\omega}_{g*}(1) = 0,$
2. $\mu^{\omega}_{g*}(a^{-1}) = -\mu^{\omega}_{g*}(a),$
3. $\mu^{(f^{-1})^{*}\omega}_{g*}(faf^{-1}) = \mu^{\omega}_{g*}(a),$
4. $sign_g = \delta\mu^{\omega}_{g*}$ *on* Δ^{ω}_{g*}.

The main property in this lemma is 4. In order to prove it, we need the following theorem proved by Atiyah, Patodi and Singer.

Theorem 9 (Atiyah-Patodi-Singer [2]). *Let M be a closed oriented manifold of odd dimension and $\alpha \colon \pi_1(M) \to U(n)$ a unitary representation. If M is the boundary of a compact oriented manifold N with α extending to a unitary representation of $\pi_1(N)$ then $\rho_{\alpha}(M) = n\ sign(N) - sign_{\alpha}(N)$.*

We consider the Σ_g-bundle $N_{a,b}$ over P, where $a, b \in \mathcal{M}^{\omega}_{g*}$. There is a unique homomorphism $\omega_{a,b} \colon \pi_1(N_{a,b}) \to \mathbb{Z}_2 \subset U(1)$ satisfying the same condition as ω_a. We apply Atiyah-Patodi-Singer's theorem to the pair $(N_{a,b}, \omega_{a,b})$ and use the Leray-Serre spectral sequence of the fibration $N_{a,b} \to P$. Then we have the property 4 in Lemma 8. Using Lemma 8, it is easy to see that the function μ^{ω}_{g*} descends to a function μ^{ω}_{g} on $\Delta^{\omega}_{g} := j(\Delta^{\omega}_{g*})$ for any $\omega \in H'$.

Theorem 10. *The difference $\phi_g - \mu_g^\omega$ is a nontrivial homomorphism from Δ_g^ω to $\mathbb{Q}$ for any $\omega \in H'$ and the equality $\phi_g = \frac{1}{\sharp H'} \sum_{\omega \in H'} \mu_g^\omega$ holds on $\mathcal{H}_g^\sigma$ for $g = 1, 2$ and 3.*

Since the Meyer function ϕ_g has the same properties as those in Lemma 8, the former part of this theorem follows from Lemma 8 and nontrivial examples which can be given explicitly. The latter follows from Lemma 8 and $H^1(\mathcal{H}_g^\sigma, \mathbb{Q})^{\mathfrak{S}_{2g+2}} = \{0\}$ which is obtained from the fact of $H^1(\mathcal{H}_g, \mathbb{Q}) = \{0\}$ using the Hochschild-Leray-Serre spectral sequence of the short exact sequence $1 \to \mathcal{H}_g^\sigma \to \mathcal{H}_g \to \mathfrak{S}_{2g+2} \to 1$.

References

[1] M. F. Atiyah, *The logarithm of the Dedekind η-function*, Math. Ann. 278(1987), 335-380.

[2] M. F. Atiyah, V. K. Patodi and I. M. Singer, *Spectral asymmetry and Riemannian geometry II*, Math. Proc. Camb. Phil. Soc. 78(1975), 405-432.

[3] J. Birman and H. Hilden, *On the mapping class groups of closed surfaces as covering spaces*, in: Advances in the Theory of Riemann Surfaces, Ann. of Math. Stud. 66(1971), 81-115.

[4] H. Endo, *Meyer's signature cocycle and hyperelliptic fibrations*, Math. Ann. 316(2000), 237-257.

[5] R. Kasagawa, *On a function on the mapping class group of a surface of genus* 2, to appear in Topology Appl.

[6] R. Kasagawa, *ρ-invariants, signature cocycles and the mapping class group of a surface*, preprint.

[7] R. Kirby and P. Melvin, *Dedekind sums, μ-invariants and the signature cocycle*, Math. Ann. 299(1994), 231-267.

[8] E. Looijenga, *Prym representations of mapping class groups*, Geom. Dedicata 64(1997), 69-83.

[9] Y. Matsumoto, *Lefshetz fibrations of genus two; -a topological approach-*, in Proceedings of the 37th Taniguchi symposium on topology and Teichmüller spaces, ed. by Sadayoshi Kojima et al. 1996, World Scientific Publishing Co. pp123-148.

[10] W. Meyer, *Die Signatur von Flächenbündeln*, Math. Ann. 201(1973), 239-264.

[11] T. Morifuji, *On Meyer's function of hyperelliptic mapping class groups*, preprint, 1998.

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